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Newfound Research White Paper

Allocating Under Uncertainty: Simple Heuristics & Complex Models

Newfound Research LLC
March 22, 2013

Abstract

In this paper we compare optimal solutions for maximum Sharpe ratio portfolios, minimum risk portfolios, maximum expected excess return portfolios, and the maximally diversified portfolios against naïve, equal-weight proxies. We find that in all cases of greater uncertainty, a more complex methodology does not provide any excess benefit over the implementations based on simple heuristics. In cases where parameters exhibit short-term stationarity, more complicated optimization methodologies can provide a benefit. In all cases, the methodology that had the least reliance on the parameters that exhibited the least rank-stability typically outperformed; when both methods relied on parameters with the same degree of rank instability, there was no solid evidence to the benefit of either strategy.

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Introduction

Modern portfolio theory ("MPT") is a theory developed in the 1950s by Harry Markowitz that attempts to maximize portfolio expected return for a given level of risk, or minimize risk for a given expected return. While diversification is by no means a modern concept, MPT provides a mathematical formulation of the concept, allowing the combination of risky assets into less-risky portfolios. The most efficient of these portfolios (e.g. those with the highest expected return for a given risk level) fall along the Efficient Frontier and can be found through mean-variance optimization ("MVO").

Despite formulating this mathematically efficient investment theory, Harry Markowitz himself has been quoted as saying that he did not follow it for his own portfolio: "I should have computed the historical co-variances of the asset classes and drawn an efficient frontier. Instead, I visualized my grief if the stock market went way up and I wasn't in it -- or if it went way down and I was completely in it. My intention was to minimize my future regret. So I split my contributions 50/50 between bonds and equities" (Zweig 2007).

Modern portfolio theory is not without its critics, especially from the field of behavioral economics where critiques mainly center on the assumption that investors always act rationally; Markowitz may have provided the perfect example that investors, in fact, do not. Besides rationality, MPT has two other major assumptions: (i) that the market structure can be perfectly described with expected returns, volatilities, and correlations and (ii) that these parameters can be extracted from the marketplace without any uncertainty.

Consider the number of estimates required for an N-asset MVO is:

- N expected returns
- N volatilities
- $(N^2 - N)/2$ correlation coefficients

Combined, an MVO optimization requires $(N^2 + 3N)/2$ parameter estimations. In other words, if the average mutual fund holds 140 different stocks, an MVO would require 10,010 precise estimates (Money.CNN.com 2008). Any degree of uncertainty in estimates would lead to the construction of a suboptimal portfolio.

While optimizing 140 weights over 10,010 potentially ill-defined estimates may be computationally intractable, certainly such a problem should achievable for Markowitz's stock-bond two-asset case, where only 5 parameters need to be estimated. The key assumption, however, in MPT is that these parameters can be estimated without any uncertainty. As a counter-point to the reality of this assumption, consider that realized 252-day daily return correlation between the SPDR S&P 500 ETF SPY and the iShares Barclays Aggregate Bond Fund AGG:

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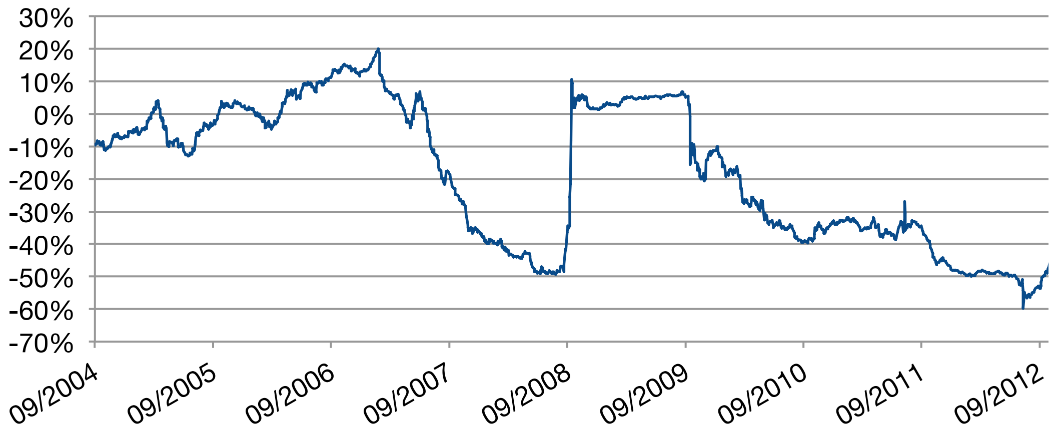

The only certainty is that the one-year realized correlation has been entirely uncertain.

A Mathematical Intuition of MVO

Consider the case where the portfolio is constructed from N assets whose returns are independent from one-another. This reduces the number of estimated parameters down to 2N (expected returns and volatilities) since independence implies zero correlation. Mathematically, the *unconstrained* solution to Sharpe-maximizing mean-variance optimization for this portfolio is (assuming a zero risk-free rate):

$$w = \frac{1}{1^T \Sigma^{-1} \mu} \Sigma^{-1} \mu$$

Where Σ is the covariance matrix and μ is the expected returns. Since the covariance matrix is a diagonal matrix (i.e. has all 0s on the non-diagonal element since correlations are zero), its inverse is trivially calculated by replacing the diagonal elements with their reciprocals.

Therefore, the term $\Sigma^{-1}\mu$ computes a vector of expected returns divided by asset variance; the term $1^T \Sigma^{-1} \mu$ sums those values. In other words, the weight for a given asset is equal to its proportional $\frac{\mu}{\sigma^2}$. Another way to think about this solution is that expected returns are first leveraged (or de-leveraged) such that asset variances are all equivalent and weights are equal to proportional leveraged (or de-leveraged) expected returns.

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At first blush, this mathematical exercise seems to be no more than a trivial solution to an unrealistic scenario since asset returns are taken to be independent. When sophisticated statistical techniques are applied, however, it provides tremendous insight and intuition into mean-variance solutions. Consider a technique called "principal component analysis," which will take an NxN variance-covariance matrix and return the asset weights that will construct N independent *principal portfolios*. The general intuition for these *principal portfolios* is that they represent different risk factors (e.g. 'market risk' or 'interest rate risk') or highly correlated asset clusters (e.g. European countries or commodities).

Explaining this statistical decomposition is beyond the scope of this document, but assuming such a decomposition exists, it makes the trivial solution and intuition discussed above highly relevant: mean-variance optimization will (1) construct *principal portfolios* with statistically independent returns, (2) leverage these portfolios such that their variances are equivalent, then (3) set their weights equal to their relative leveraged expected returns.

Therefore, in the case where the lowest variance principal portfolios have high relative expected excess returns, they will dominate the final portfolio composition. Random Matrix Theory, a branch of probability theory related to the understanding of matrix-valued random variables, says that those very same *principal portfolios* may be entirely dominated by measurement noise, and hence may be very unstable from one estimation period to the next.

The process, therefore, is highly sensitive to the estimated parameters. MVO relies on accurate correlation estimates to construct the independent *principal portfolios*; it relies on accurate variance measures to accurately leverage assets; it relies on accurate expected returns to create proportional measures. Since this is a non-linear, dynamic process, the behavior of the process will be highly sensitive to the initial conditions, and small changes can lead to highly divergent outcomes.

The mathematics show that MVO should provide the key to unlocking the most efficient portfolios. Recognizing that the uncertainty assumption does not hold, the question becomes: how robust is MVO to the uncertainty in the parameter estimation? More applicably, do the benefits outweigh of MVO the risks?

To test this question, long-only MVO is compared against a parameter-robust allocation methodology, equal weight ("EW"), over four scenarios: the maximum Sharpe portfolio, the minimum variance (minimum risk) portfolio, the maximum

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expected excess return (maximum risk) portfolio, and the maximally diversified portfolio.

The Maximum Sharpe Ratio Portfolio

The portfolio along the efficient frontier that maximizes expected excess return per unit risk is called the "tangency portfolio" – or simply the maximum Sharpe ratio portfolio.

Test #1: Synthetic Market Structures

In the first test, different market structures (expected returns, volatilities, and correlations) are simulated under two scenarios:

1. All assets have a Sharpe ratio of 0.5
2. Asset Sharpe ratios are distributed between 0 and 0.5

For both cases, annualized expected returns for assets vary uniformly between 0% and 16%. Volatilities are derived from the relationship between Sharpe ratios and expected returns. The correlation structure for any two assets is uniformly selected between -1 and 1, and Highham's algorithm is used to move the randomly selected correlation structure to the nearest *true* correlation matrix (Higham 2002).

In the test, a 50-asset portfolio is assumed.

From the *true* market structure, samples of varying lengths (10, 25, 50, 75, 100, 250, 500, 1000, 1500, 2000, 2500, and 3000 month periods) are pulled, from which an *estimated* market structure is created. From the estimated market structure, MVO is used to determine asset weights. The MVO asset weights and an equal-weight scheme, whose allocations were based upon the estimated market structure, are then invested (over a 36 period realized path) over the *true* market structure. The expected return and variance for each strategy over this 36-period horizon provides the basis for a single-period Sharpe ratio. This process is performed 5000 times per sample length, providing a distribution of possible Sharpe ratios for each sample length.

As an added layer of complexity, trials where more than one true market structure existed were used. In those cases, before sampling, the market structure in question was chosen randomly. So in the case of three market structures, one structure would be randomly chosen for the MVO process to construct its estimate of, and another (potentially the same) structure would be randomly chosen for the forward walk for testing the weights.

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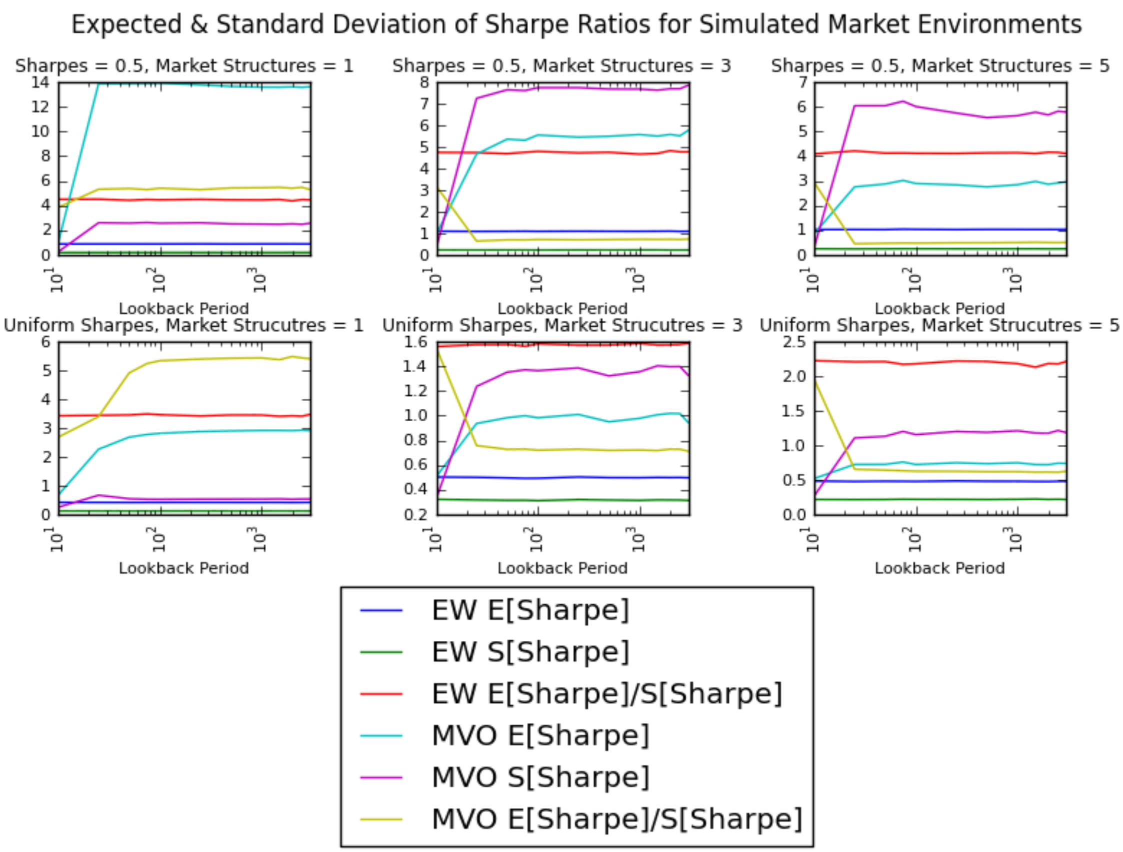
The results are summarized in the following figure:



The plots in the left-most column display the results expected from MVO in the case where a single, true market structure exists: as sample size increases, the estimated market structure should approach the true market structure, and MVO should provide the portfolio that maximizes expected Sharpe.

However, as the number of potential market structures increases, the variance in MVO Sharpe ratios increases much more rapidly than the expected MVO Sharpe values. EW, on the other hand, remains robust, no matter the number of market structures introduced, because there is no risk in the weights being mis-calibrated. The MVO expected Sharpe also approaches the EW expected Sharpe as the number of market structures increases. In other words, if there is no stability in the market structure, or the future market structure cannot be estimated from past samples, there is little to no benefit (in fact, there may be harm from increased uncertainty) from using a complex method over a simple heuristic.

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This test, however, uses completely simulated results and environments. Most unrealistic is the methodology used to switch market regimes; regimes are unlikely to instantaneously switch, but rather gradually change over several months. Therefore it is important to ask how these two methods fare with *real market data*? How do they fare when the more realistic method of *walk-forward* optimization is used?

Test #2: Bootstrapped Asset Returns

In the following test, returns from the current S&P 500 constituents[1] are used. For varying sample lengths (10, 25, 50, 75, 100, 250, 500, 1000, 1500, 2000, 2500, and 3000 month periods), fifty stocks are chosen at random, creating one of 3.44×10^{66} possible portfolios. A resampled block bootstrapping methodology is utilized to construct a 5000-month time-series for each of the 50 assets. The sample length is used to determine how much historical information to utilize in estimating the market structure, from which MVO weights are derived and used to determine portfolio investment over the next month. For each sample length, 100 trials are performed.

The following figure shows scatter plots of the 100 trials for each sample length. The plots are fit with a 2nd-degree polynomial.

[1] Technically, due to data constraints, only 442 of the current 500 constituents are utilized

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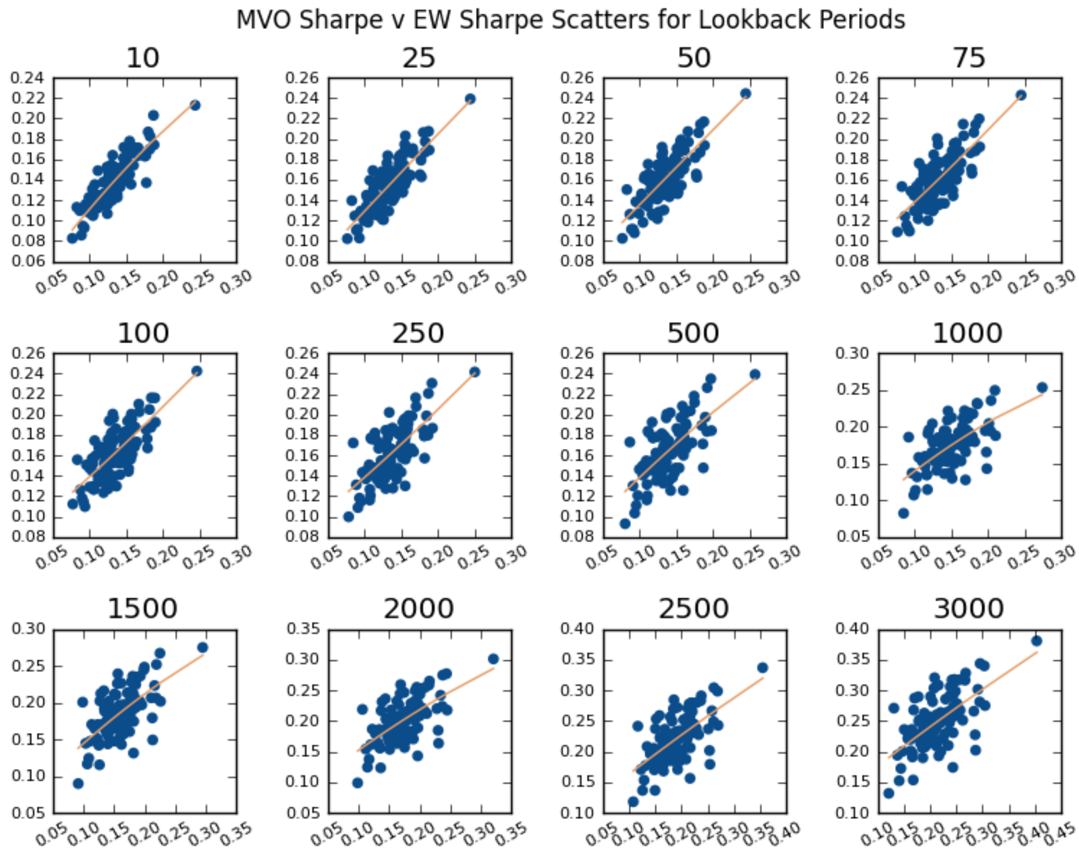

The following table summarizes the frequency with which MVO beat EW for each of the look-back periods:

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Sample Length	% MVO Sharpe > EW Sharpe
10	65%
25	92%
50	92%
75	94%
100	93%
250	89%
500	88%
1000	82%
1500	83%
2000	85%
2500	83%
3000	84%

Below are two tables summarizing the data:

	EW Sharpe Distribution					MVO Sharpe Distribution				
	Min	**5%**	**Mean**	**95%**	**Max**	**Min**	**5%**	**Mean**	**95%**	**Max**
10	0.08	0.09	0.13	0.18	0.24	0.08	0.11	0.14	0.17	0.21
25	0.08	0.09	0.13	0.18	0.24	0.10	0.12	0.16	0.19	0.24
50	0.08	0.09	0.13	0.18	0.24	0.10	0.12	0.16	0.20	0.24
75	0.08	0.09	0.14	0.18	0.25	0.11	0.12	0.16	0.20	0.24
100	0.08	0.09	0.14	0.18	0.25	0.11	0.13	0.16	0.20	0.24
250	0.08	0.09	0.14	0.18	0.25	0.10	0.12	0.16	0.20	0.24
500	0.08	0.10	0.14	0.19	0.26	0.09	0.12	0.17	0.21	0.24
1000	0.09	0.10	0.15	0.20	0.27	0.08	0.13	0.17	0.22	0.25
1500	0.09	0.11	0.16	0.22	0.29	0.09	0.14	0.19	0.24	0.27
2000	0.10	0.12	0.17	0.23	0.32	0.10	0.15	0.20	0.26	0.30
2500	0.11	0.13	0.19	0.26	0.36	0.12	0.17	0.22	0.28	0.34
3000	0.12	0.15	0.22	0.29	0.40	0.13	0.19	0.25	0.32	0.38

	Confidence Intervals of EW E[Sharpe]					Confidence Intervals of MVO E[Sharpe]				
	1%	**5%**	**Mean**	**95%**	**99%**	**1%**	**5%**	**Mean**	**95%**	**99%**
10	0.13	0.13	0.13	0.14	0.14	0.13	0.13	0.14	0.14	0.14
25	0.13	0.13	0.13	0.14	0.14	0.15	0.15	**0.16**	0.16	0.16
50	0.13	0.13	0.13	0.14	0.14	0.15	0.16	**0.16**	0.17	0.17

75	0.13	0.13	0.14	0.14	0.14	0.16	0.16	**0.16**	0.17	0.17
100	0.13	0.13	0.14	0.14	0.14	0.16	0.16	**0.16**	0.17	0.17
250	0.13	0.13	0.14	0.14	0.14	0.16	0.16	**0.16**	0.17	0.17
500	0.13	0.14	0.14	0.15	0.15	0.16	0.16	**0.17**	0.17	0.17
1000	0.14	0.15	0.15	0.16	0.16	0.17	0.17	**0.17**	0.18	0.18
1500	0.15	0.16	0.16	0.17	0.17	0.18	0.18	**0.19**	0.19	0.19
2000	0.17	0.17	0.17	0.18	0.18	0.19	0.20	**0.20**	0.21	0.21
2500	0.18	0.19	0.19	0.20	0.20	0.21	0.22	**0.22**	0.23	0.23
3000	0.21	0.21	0.22	0.22	0.23	0.24	0.24	**0.25**	0.26	0.26

Bold for one mean Sharpe indicates greater than the other mean Sharpe with 95% confidence; an underline indicates 99% confidence.

The MVO expected Sharpe is, in fact, higher at 95% and 99% confidence levels for the majority of sample lengths. However, does the higher level of expected Sharpe require a higher degree of uncertainty? The following table shows the expected Sharpe level relative to the standard deviation of the Sharpe ratios achieved:

	EW E[Sharpe] / S[Sharpe]	MVO E[Sharpe] / S[Sharpe]
10	4.97	5.73
25	4.97	6.23
50	4.97	6.35
75	4.97	6.44
100	4.97	6.51
250	4.97	6.21
500	4.96	6.07
1000	4.94	5.72
1500	4.92	5.64
2000	4.90	5.70
2500	4.86	5.73
3000	4.80	5.68

It would seem that not only is the expected Sharpe from MVO greater than the expected Sharpe of EW at 95% and 99% confidence levels for look-back periods greater than or equal to 25 months, but it is achieved with less uncertainty as well!

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Again, however, this is a simulated test: while the returns used are sampled from real market returns, the organization of the returns is simulated. The returns themselves are possible, but not necessarily the order in which they were placed.

Block bootstrapping is designed to capture the salient features of markets such as autocorrelation, leptokurtosis, micro-correlation storms and clustered volatility. While it is effective at creating new return streams for asset classes, it is ineffective at creating new market environments as measured by relative Sharpe ratios. The random block selection process can actually create an *advantage* for MVO if the selected blocks represent the different environments over the decade, providing the MVO process with insight into the potential environments it may see into the future.

Consider that market environments typically occur in a serial fashion: one environment after another. The danger for MVO is that the next environment will look nothing like the last from which estimates are derived. Block bootstrapping actually shuffles these environments, meaning that the look-back samples will provide good information about future potential environments that the portfolio will encounter; future environments will not be a single environment, but a mix of environments. Instead of the portfolio moving through several heterogeneous environments in a linear fashion, block bootstrapping creates a single homogenous environment: a perfect situation for MVO to succeed.

This test, therefore, echoes the success of MVO in the first test: when the future is similar enough to the past, MVO will maximize the expected Sharpe ratio and frequently outperform EW.

Test #3: Bootstrapped Systematic & Idiosyncratic Returns

Another critical problem with naïve block bootstrapping, when used for the construction of new market environments, is that idiosyncratic risk, which should be partially independent between assets, becomes tied together since a given block contains returns for all assets. Therefore, to allow float of idiosyncratic risks (which should allow for variation in relative Sharpe ratios), this test first finds the beta and idiosyncratic risk factor for each security, bootstraps returns from the S&P 500 index and then bootstraps the idiosyncratic risk factor. This allows each asset to share systematic risk, but generate idiosyncratic risk independently, creating new market regimes as measured by relative Sharpe ratios.

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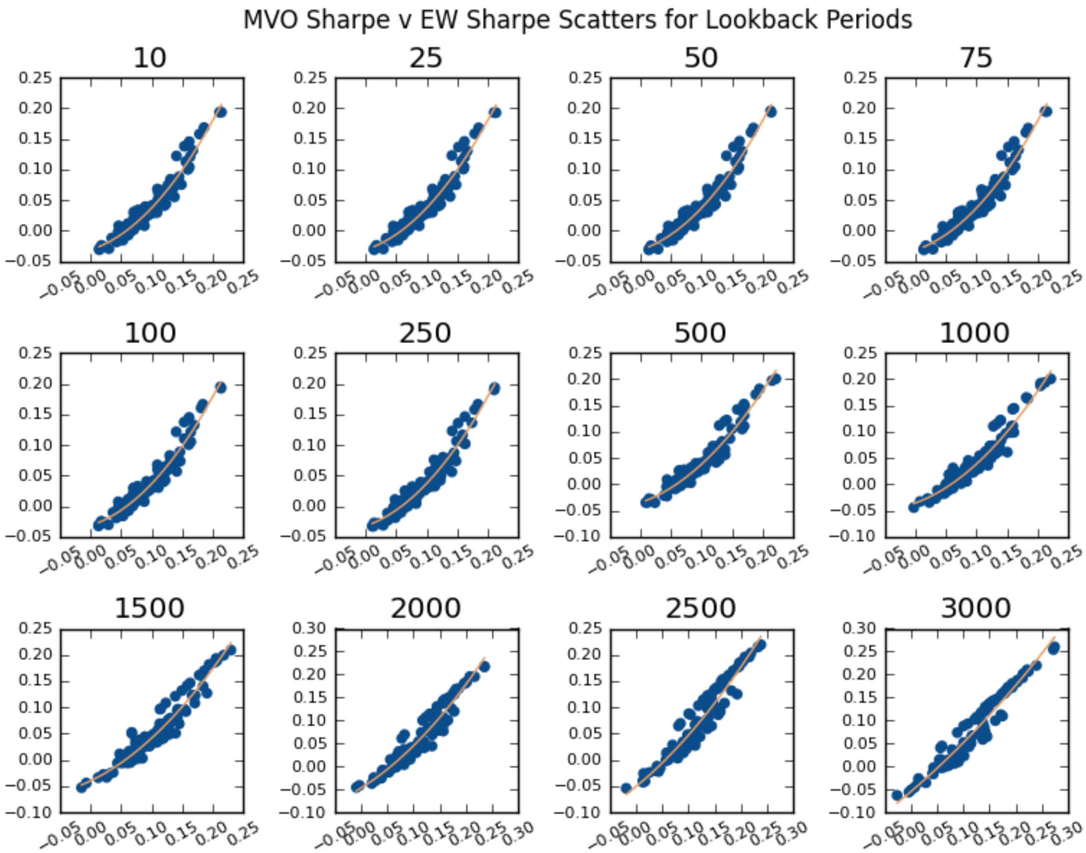

Sample Length	% MVO Sharpe > EW Sharpe
10	0%
25	0%
50	0%
75	0%
100	0%
250	0%
500	0%
1000	0%
1500	0%
2000	0%
2500	0%
3000	0%

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For one hundred unique portfolios over different look-back periods, MVO did not beat EW even once. Below are two tables summarizing the data:

	EW Sharpe Distribution					MVO Sharpe Distribution				
	Min	**5%**	**Mean**	**95%**	**Max**	**Min**	**5%**	**Mean**	**95%**	**Max**
10	0.01	0.03	0.10	0.16	0.21	-0.03	-0.02	0.04	0.14	0.19
25	0.01	0.03	0.10	0.16	0.21	-0.03	-0.02	0.04	0.14	0.19
50	0.01	0.03	0.10	0.16	0.21	-0.03	-0.02	0.04	0.14	0.20
75	0.01	0.03	0.10	0.16	0.22	-0.03	-0.02	0.04	0.14	0.19
100	0.01	0.04	0.10	0.16	0.21	-0.03	-0.02	0.04	0.14	0.19
250	0.01	0.04	0.10	0.16	0.21	-0.03	-0.02	0.04	0.14	0.19
500	0.01	0.04	0.10	0.17	0.22	-0.04	-0.02	0.04	0.15	0.20
1000	0.00	0.03	0.10	0.18	0.22	-0.04	-0.03	0.05	0.16	0.20
1500	-0.01	0.03	0.11	0.19	0.23	-0.05	-0.03	0.05	0.18	0.21
2000	-0.01	0.04	0.11	0.20	0.24	-0.05	-0.02	0.06	0.18	0.22
2500	-0.02	0.02	0.12	0.20	0.24	-0.05	-0.02	0.07	0.19	0.22
3000	-0.03	0.01	0.12	0.22	0.27	-0.06	-0.03	0.08	0.21	0.26

	Confidence Intervals of EW E[Sharpe]					Confidence Intervals of MVO E[Sharpe]				
	1%	**5%**	**Mean**	**95%**	**99%**	**1%**	**5%**	**Mean**	**95%**	**99%**
10	0.09	0.09	**0.10**	0.10	0.11	0.03	0.03	0.04	0.05	0.05
25	0.09	0.09	**0.10**	0.10	0.11	0.03	0.03	0.04	0.05	0.05
50	0.09	0.09	**0.10**	0.10	0.11	0.03	0.03	0.04	0.05	0.05
75	0.09	0.09	**0.10**	0.10	0.11	0.03	0.03	0.04	0.05	0.05
100	0.09	0.09	**0.10**	0.10	0.11	0.03	0.03	0.04	0.05	0.05
250	0.09	0.09	**0.10**	0.10	0.11	0.03	0.03	0.04	0.05	0.05
500	0.09	0.09	**0.10**	0.10	0.11	0.03	0.03	0.04	0.05	0.05
1000	0.09	0.09	**0.10**	0.11	0.11	0.03	0.04	0.05	0.05	0.06
1500	0.09	0.10	**0.11**	0.11	0.12	0.04	0.04	0.05	0.06	0.07
2000	0.10	0.10	**0.11**	0.12	0.12	0.05	0.05	0.06	0.07	0.08
2500	0.10	0.11	**0.12**	0.12	0.13	0.05	0.06	0.07	0.08	0.09
3000	0.10	0.11	**0.12**	0.13	0.13	0.06	0.06	0.08	0.09	0.10

Bold for one mean Sharpe indicates greater than the other mean Sharpe with 95% confidence; an underline indicates 99% confidence.

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Not surprisingly, given the frequency with which EW beat MVO, expected Sharpe for the EW process is greater than MVO expected Sharpe for every look-back period with 99% confidence.

The following table shows the expected Sharpe level relative to the standard deviation in the Sharpe achieved:

	EW E[Sharpe] / S[Sharpe]	MVO E[Sharpe] / S[Sharpe]
10	2.37	0.85
25	2.37	0.85
50	2.36	0.85
75	2.35	0.84
100	2.35	0.84
250	2.32	0.83
500	2.24	0.82
1000	2.21	0.85
1500	2.12	0.92
2000	2.15	0.99
2500	2.09	1.02
3000	1.89	0.99

While the results are astoundingly promising for EW, it is potentially because the process introduces *too much* uncertainty into the marketplace; in the simulation, market regimes may change at an unrealistically rapid pace making it impossible for MVO to calibrate. Furthermore, the process makes idiosyncratic risk for each security entirely independent, which is also likely unrealistic. So while Test #2 may create too much homogeneity, Test #3 arguably introduces too much heterogeneity to be realistic.

Test #4

To test in a real market environment, the same process for selecting portfolios is utilized as test #2, but the bootstrap is replaced with the actual security performances over their shared history, the look-back periods are reduced (to 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, and 120 month periods), and the number of portfolios is increased to 1000 per look-back period. Each portfolio begins at the earliest possible starting date, depending on the securities selected, and runs through to present.

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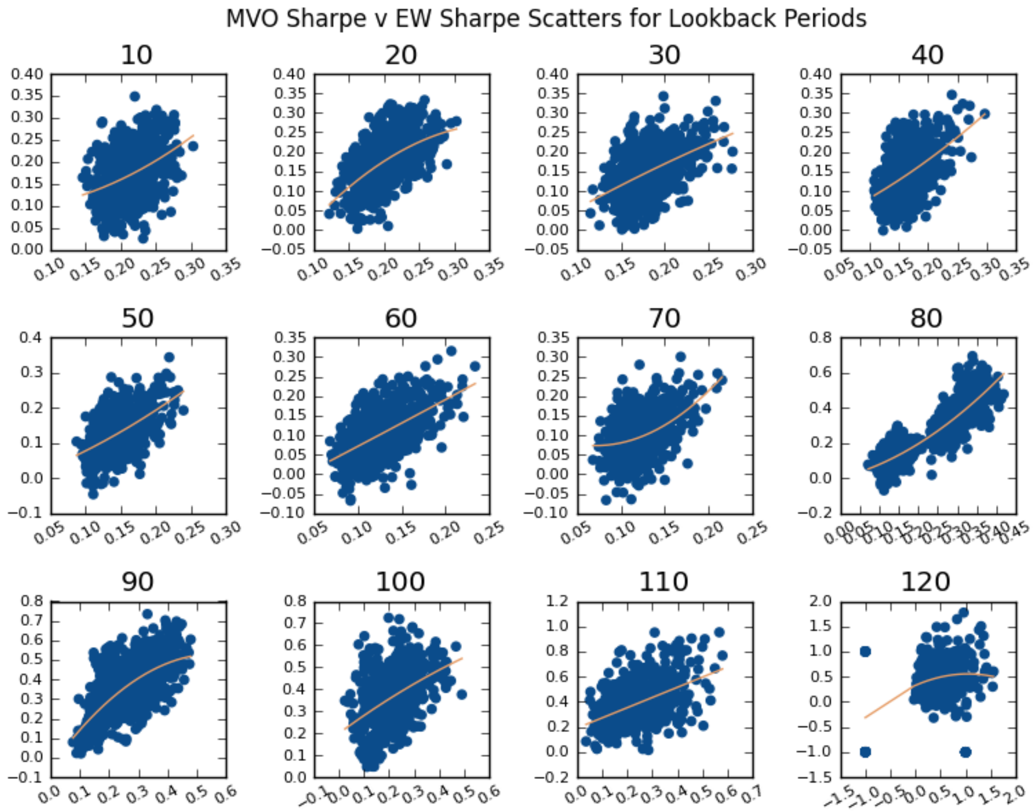
The following figure shows a scatter plot of the 1000 trials per sample-size. Again, a 2nd-degree polynomial is fit.



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The results are far less clear than before.

Sample Length	% MVO Sharpe > EW Sharpe
10	20.87%
20	28.50%
30	25.45%
40	33.72%
50	34.99%
60	32.57%
70	30.79%
80	48.85%
90	78.88%
100	91.09%
110	90.33%
120	68.45%

Interestingly, the results seem to be almost the exact opposite of our first test.

	EW Sharpe Distribution					MVO Sharpe Distribution				
	Min	-95% CI	Mean	95% CI	Max	Min	-95% CI	Mean	95% CI	Max
10	0.15	0.17	0.21	0.26	0.30	0.03	0.08	0.17	0.26	0.35
20	0.12	0.16	0.20	0.26	0.30	0.00	0.08	0.18	0.28	0.33
30	0.12	0.15	0.18	0.23	0.28	0.00	0.06	0.15	0.24	0.34
40	0.11	0.13	0.17	0.22	0.30	0.00	0.05	0.14	0.24	0.35
50	0.09	0.11	0.15	0.19	0.24	-0.05	0.04	0.13	0.23	0.34
60	0.07	0.09	0.13	0.18	0.23	-0.07	0.02	0.11	0.21	0.32
70	0.07	0.09	0.12	0.16	0.22	-0.07	0.02	0.10	0.19	0.30
80	0.07	0.11	0.21	0.36	0.42	-0.07	0.05	0.22	0.52	0.69
90	0.08	0.12	0.26	0.41	0.48	0.02	0.10	0.35	0.58	0.73
100	0.02	0.10	0.22	0.33	0.49	0.05	0.14	0.37	0.55	0.72
110	0.04	0.10	0.23	0.42	0.58	0.02	0.16	0.38	0.65	0.95
120	-1.00	0.02	0.36	1.00	1.56	-1.00	-0.83	0.42	1.00	1.77

	Confidence Intervals of Mean EW Sharpe					Confidence Intervals of Mean MVO Sharpe				
	-99%	95%	Mean	95%	99%	-99%	-95%	Mean	95%	99%
10	0.21	0.21	**0.21**	0.22	0.22	0.16	0.16	0.17	0.18	0.18
20	0.20	0.20	**0.20**	0.21	0.21	0.16	0.17	0.18	0.19	0.19

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30	0.18	0.18	**0.18**	0.19	0.19	0.14	0.14	0.15	0.16	0.16
40	0.16	0.16	**0.17**	0.17	0.17	0.13	0.13	0.14	0.15	0.16
50	0.14	0.14	**0.15**	0.15	0.15	0.12	0.12	0.13	0.14	0.14
60	0.12	0.13	**0.13**	0.13	0.14	0.09	0.10	0.11	0.12	0.12
70	0.12	0.12	**0.12**	0.13	0.13	0.09	0.09	0.10	0.11	0.11
80	0.19	0.19	0.21	0.22	0.23	0.18	0.20	0.22	0.25	0.26
90	0.24	0.25	0.26	0.28	0.28	0.32	0.33	**0.35**	0.38	0.39
100	0.20	0.21	0.22	0.23	0.24	0.34	0.35	**0.37**	0.39	0.40
110	0.21	0.22	0.23	0.25	0.25	0.34	0.35	**0.38**	0.40	0.41
120	0.26	0.29	0.36	0.43	0.46	0.32	0.35	0.42	0.49	0.52

Bold for one mean Sharpe indicates greater than the other mean Sharpe with 95% confidence; an underline indicates 99% confidence.

The following table shows the expected Sharpe level relative to the uncertainty in the Sharpe that will be achieved:

	EW E[Sharpe] / S[Sharpe]	MVO E[Sharpe] / S[Sharpe]
10	8.18	3.09
20	6.67	2.96
30	7.18	2.65
40	5.90	2.49
50	5.86	2.29
60	4.70	1.84
70	5.47	1.87
80	2.28	1.44
90	3.02	2.51
100	3.13	3.12
110	2.38	2.52
120	0.85	0.97

While we see that longer-samples seem to favor MVO, when portfolio data length is taken into consideration, the majority of portfolios selected will only have 120 months in length. Therefore, while 10-month look-back tests have the opportunity to walk forward 110 months, a portfolio with a 120-month look-back may only have the opportunity to walk forward a couple of months. The following table shows the average walk-forward length for each sample-period length.

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Sample Length	Mean Walk-Forward Length
10	123
20	113
30	103
40	93
50	83
60	73
70	63
80	53
90	43
100	33
110	23
120	13

On average, those tests for 110 and 120-month look-back periods only had 23 and 13 monthly returns, respectively, to generate the Sharpe ratios. The question that must be asked is whether 23 and 13 monthly returns are enough to generate accurate Sharpe ratios, which rely on an accurate measure of expected returns, knowing that the general rule of thumb for invoking the Law of Large Numbers is a sample size in excess of 30.

Limitations on Tests #2, #3 and #4

Consider the following graph, which plots the availability of data:

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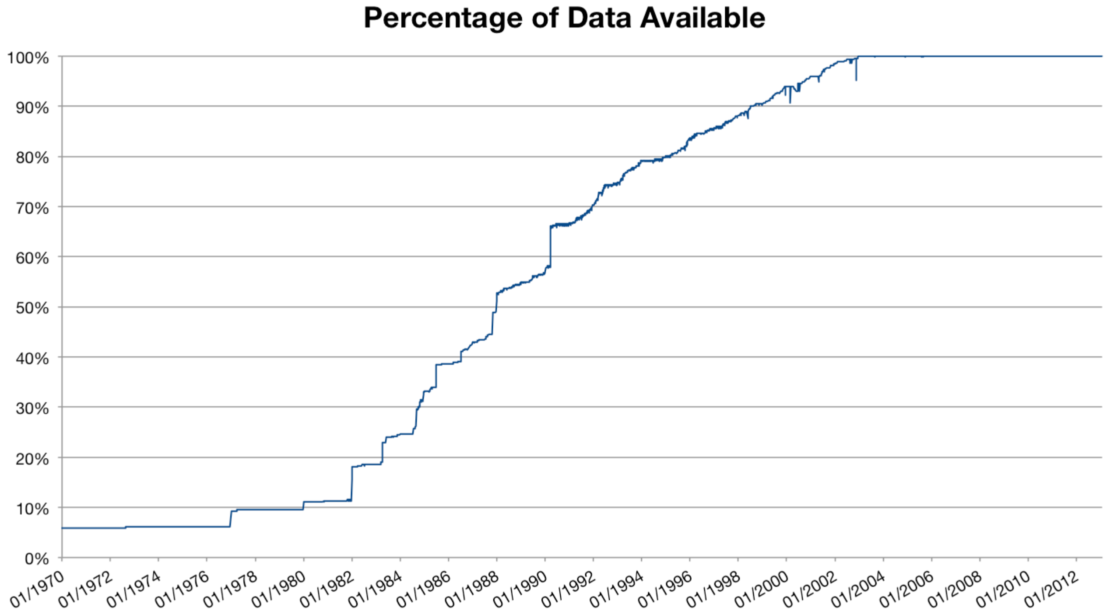

Note that the graph should be monotonically increasing; any dips in the data represent dates where securities have a gap in their data.

A uniform selection of 50 equities from the available 442 means that each equity has an initial 11.31% chance of being selected. The probability, therefore, of selecting a portfolio that has more than three decades of available information means that the randomly selected 50 stocks must come from a selection of 82 of the 442 available equities: a 1 in 4.3×10^{44} chance event. Even choosing 50 equities with data from the last 15 years is a 1 in 1116 chance. The graph below plots the probability of selecting a portfolio of equities with at least data going back to the given date; the graph indicates there is a 50% chance that the portfolio constructed will contain data from only 2002 and onward.

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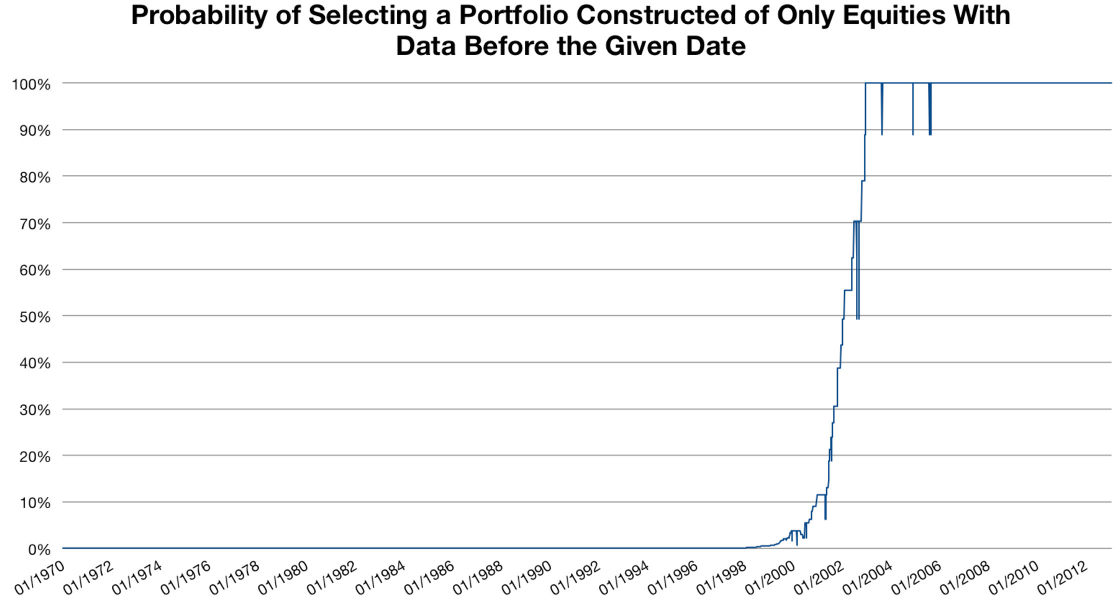

Note that the graph should be monotonically increasing; any dips in the data represent dates where securities have a gap in their data.

Even if we reduce the number of assets in the portfolio to 10, the picture doesn't look much better:

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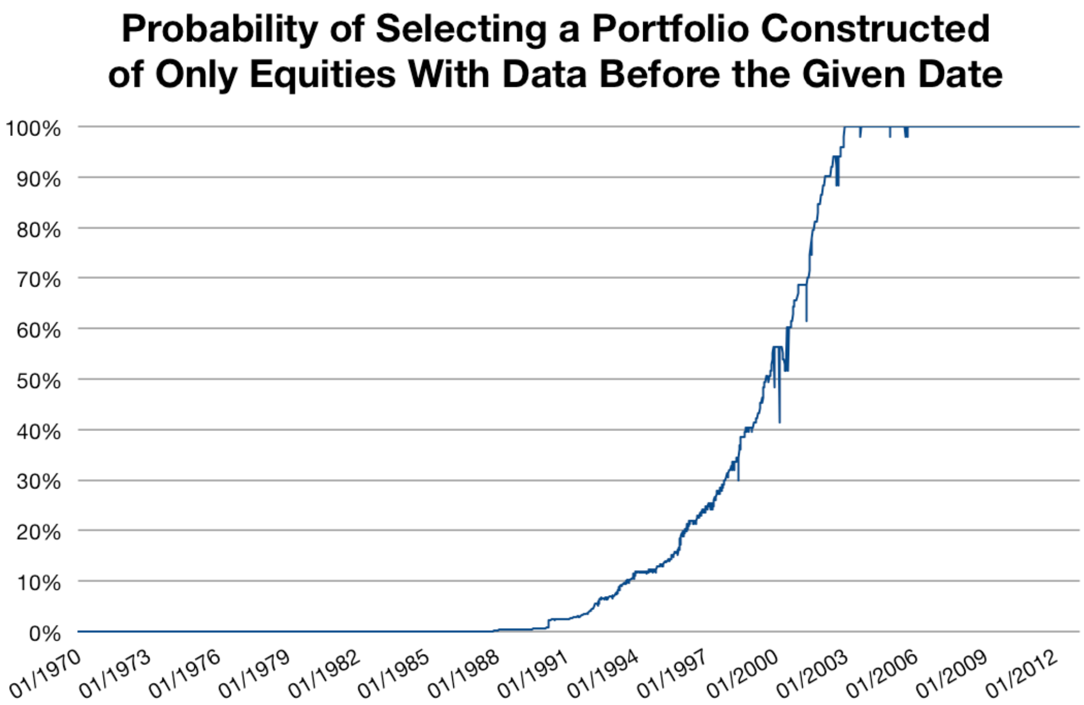

Note that the graph should be monotonically increasing; any dips in the data represent dates where securities have a gap in their data.

There is a 50% chance that the portfolio selected will contain data only from August 1999 onward. Therefore, while the tests explore a variety of different portfolios, it is likely that they will all contain data generated from the same market environments.

Concluding Remarks on Constructing the Maximum Sharpe Ratio Portfolio

The four tests demonstrate that when there is a realistic belief that the future market environment will be similar to the one over which MVO calibrates (homogeneity of environments), MVO will provide a statistically significant higher expected Sharpe ratio than EW. However, under uncertainty, EW proves to be more robust as it does not require the estimation of any parameters.

The Minimum Variance (Minimum Risk) Portfolio

Finding the maximum Sharpe ratio portfolio requires accurate estimates for expected excess returns, variances, and correlations. While a naïve EW allocation scheme exhibited higher Sharpe ratios than MVO under many

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scenarios, it may be beneficial to determine whether the cause was due to estimations of expected excess returns, which are notoriously hard to estimate.

With that said, by removing expected excess return (or, equivalently, setting all expected excess returns to 1), MVO will solve for the minimum variance portfolio: the combination of securities that, in combination, exhibits the lowest variance. In comparison, EW portfolio will simply choose the N securities with the lowest volatility.

This test captures the very real choices made by practitioners every day. The recent popularity and success of minimum-volatility portfolios has led to the development of several funds. S&P and MSCI have released two such strategies: the S&P 500 Low Volatility ETF ("SPLV") and the MSCI USA Minimum Volatility Index ETF ("USMV"). While both strategies seek to track a low volatility basket, they take very different approaches:

- S&P uses the previous year to determine volatility estimates for assets and chooses the 100 least volatile equities, rebalancing quarterly
- MSCI uses a multi-factor risk model (GEM2) to forecast volatility and uses the Barra Open Optimizer, subject to various constraints (such as sector weightings), to find the minimum-variance portfolio

A comparison of the indices that these funds track shows that from 2/27/2008 through 9/10/2012, SPLV had 18.94% annualized volatility whereas USMV had 22.13% annualized volatility.

The Test

To test whether a quadratic optimization provides an edge over a simple equal-weight solution, two portfolios are constructed:

1. Use the previous M days to develop volatility estimates and construct an EW portfolio the N assets with the lowest volatilities.
2. Use the previous M days to estimate the variance-covariance matrix and use quadratic optimization to solve for the weights that construct the minimum variance portfolio

Both portfolios rebalance monthly, using the previous M days' data to re-calibrate. By varying M (the amount of historical information to use) and N (the number of securities in the EW portfolio), two factors can be determined:

1. Is there a portfolio size threshold whereby EW or MVO methods supersede each other? How does this portfolio size compare to traditional portfolio sizes held by individual investors?

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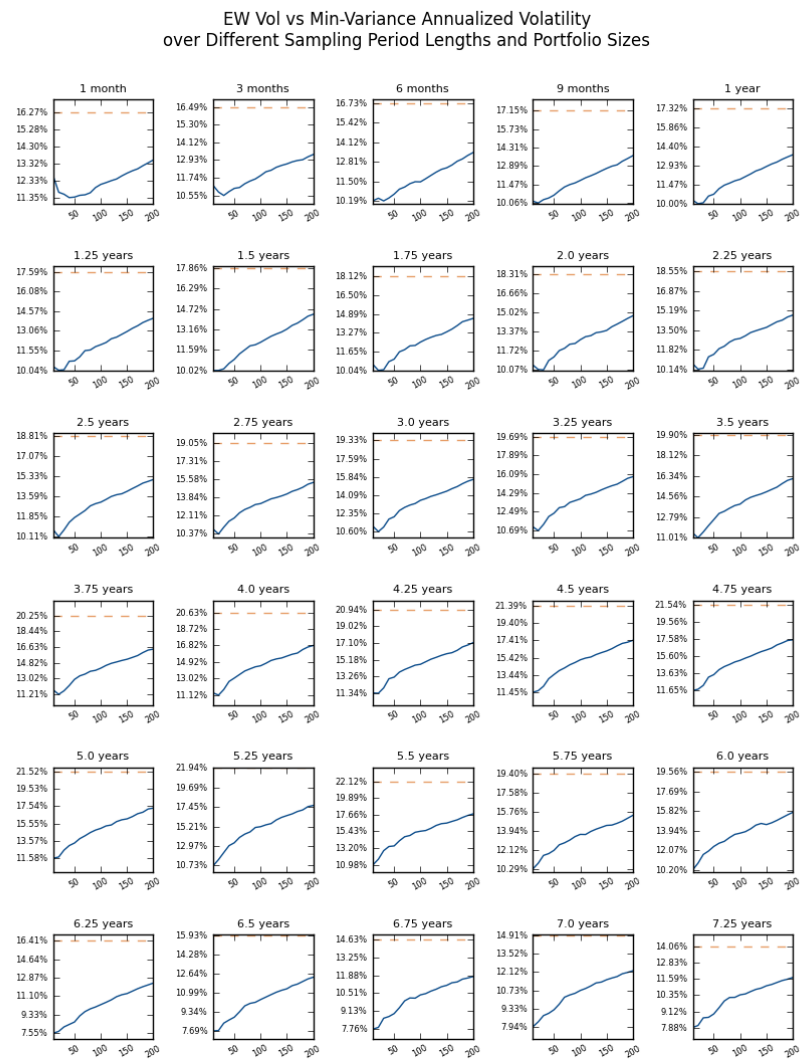
2. Is there a look-back period size required for variance-covariance matrix estimate stability that provides MVO with an edge?

The results are plotted below. For each plot, the orange, dashed line represents the MVO realized-volatility using an M-month look-back period. The blue, solid line in each plot is the realized volatility for an EW portfolio made up of N assets, where N is plotted on the x-axis.



The results are staggering: no matter the look-back period, for portfolios ranging from only 10 securities up to 200, equal-weighting the N lowest-volatility securities consistently demonstrated reduced realized volatility in comparison to the more complicated minimum-variance portfolio. This phenomenon likely

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occurs because volatility ranking is likely to be more consistent than correlation estimates, which may cause the optimization process to create highly concentrated positions into inversely correlated securities. Therefore, in the construction of the minimum variance portfolio, the correlation estimates are just as, if not more, important as the variance estimates.

While inaccuracy in estimating expected excess returns is normally labeled as the culprit in sub-optimal MVO performance, the sheer magnitude of the variance-covariance matrix in question may simply lead to the introduction of too much estimation error; for the 442 securities in question, the optimization requires the estimation of 97,461 parameters.

Test Limitations

Similar to the limitations in the maximum Sharpe ratio analysis, the walk-forward data is limited to a limited time period shared by equities. Analysis, therefore, starts on 12/26/2002. This is important for two reasons. Firstly, it means that all analysis took place over the 2002-2013 realized market cycle and does not account for other potential market cycles that were not realized. Secondly, the walk-forward period for long look-back samples is restricted; the 7.25-year look-back period is limited to only 33 walk-forward months in testing. This brings into question the statistical accuracy of the high look-back length tests.

The Maximum Expected Return (Maximum Risk) Portfolio

On the complete opposite end of the efficient frontier from the minimum variance (minimum risk) portfolio is the maximum expected return (maximum risk) portfolio. In the quadratic optimization, variance and correlation estimates are trumped in importance by expected excess return estimates; whereas the last test helps determine whether expected return may be the weak link, this test helps determine whether variance and correlation estimation error may be the cause.

The Test

To test whether a quadratic optimization provides an edge over a simple equal-weight solution, two portfolios are constructed:

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1. A momentum portfolio: use the previous M days to develop expected return estimates and construct an EW portfolio the N assets with the highest expected returns

2. Use the previous M days to estimate the variance-covariance matrix and expected returns and use quadratic optimization to solve for the weights that construct the maximum expected return portfolio

Like the previous test, both portfolios rebalance monthly, using the previous M days' data to re-calibrate. The formulation of the test allows the same questions to be answered:

1. Is there a portfolio size threshold whereby EW or MVO methods supersede each other? How does this portfolio size compare to traditional portfolio sizes held by individual investors?
2. Is there a look-back period size required for variance-covariance matrix estimate stability that provides MVO with an edge?

The results are plotted below. For each plot, the orange, dashed line represents the MVO realized annualized return using an M-month look-back period. The blue, solid line in each plot is the realized annualized return for an EW portfolio made up of N assets, where N is plotted on the x-axis.

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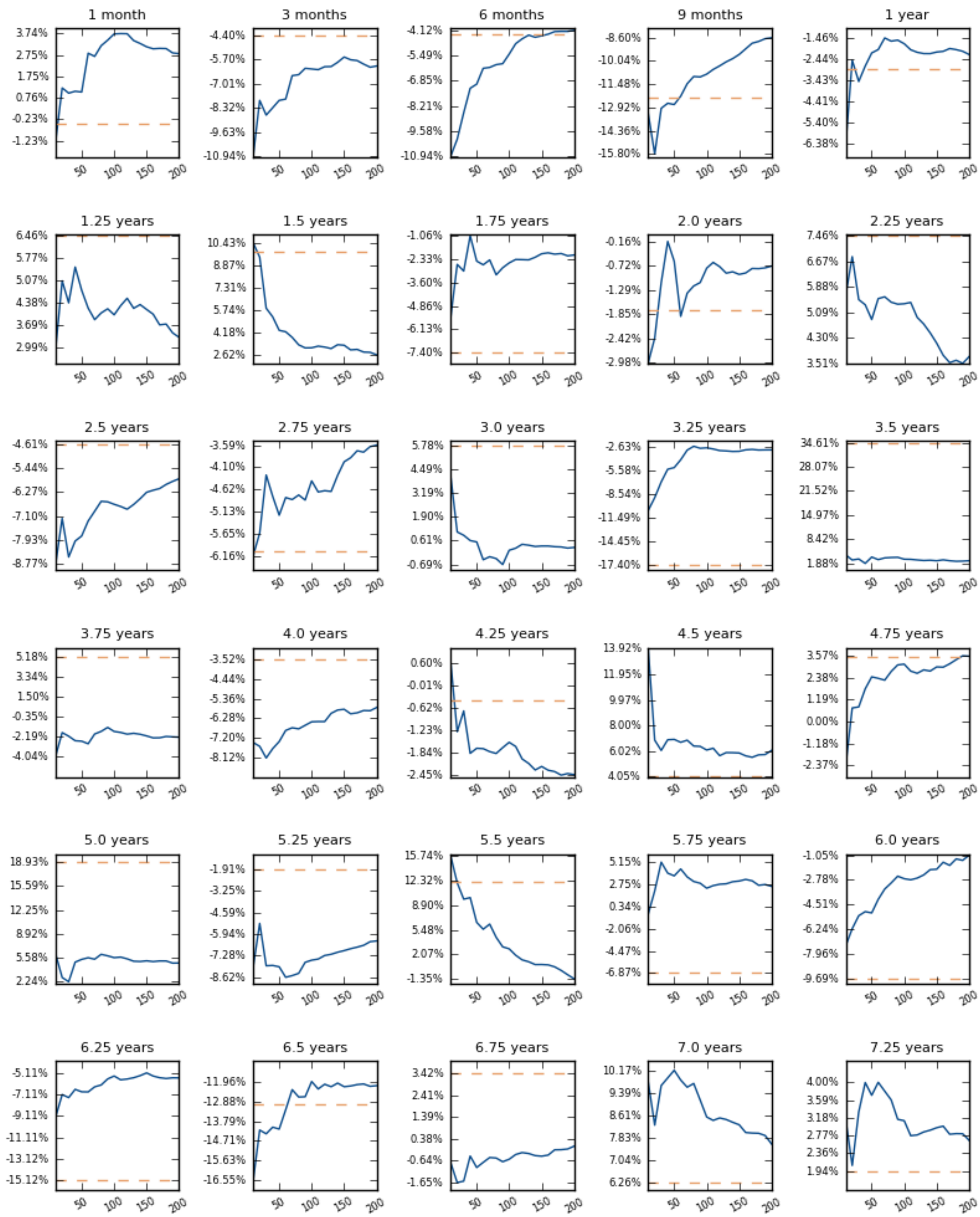
EW Annualized Return vs Max Risk Annualized Return over Different Sampling Period Lengths and Portfolio Sizes



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The results are very different than the minimum variance portfolio test; the results are split about 50/50 based on sampling period size. While the minimum variance portfolio may have introduced an exceptional amount of estimation error via the 442x442 correlation matrix, the riskiest portfolio will heavily weight the influence of the expected excess return estimations, placing far more relative emphasis on the 442 expected excess return parameters.

Unlike variance estimates, the ranking of expected excess return estimates shows a lower degree of stability. Below is a graph plotting the average rank stability for returns, volatilities, and average pairwise correlations for different look-back period lengths.

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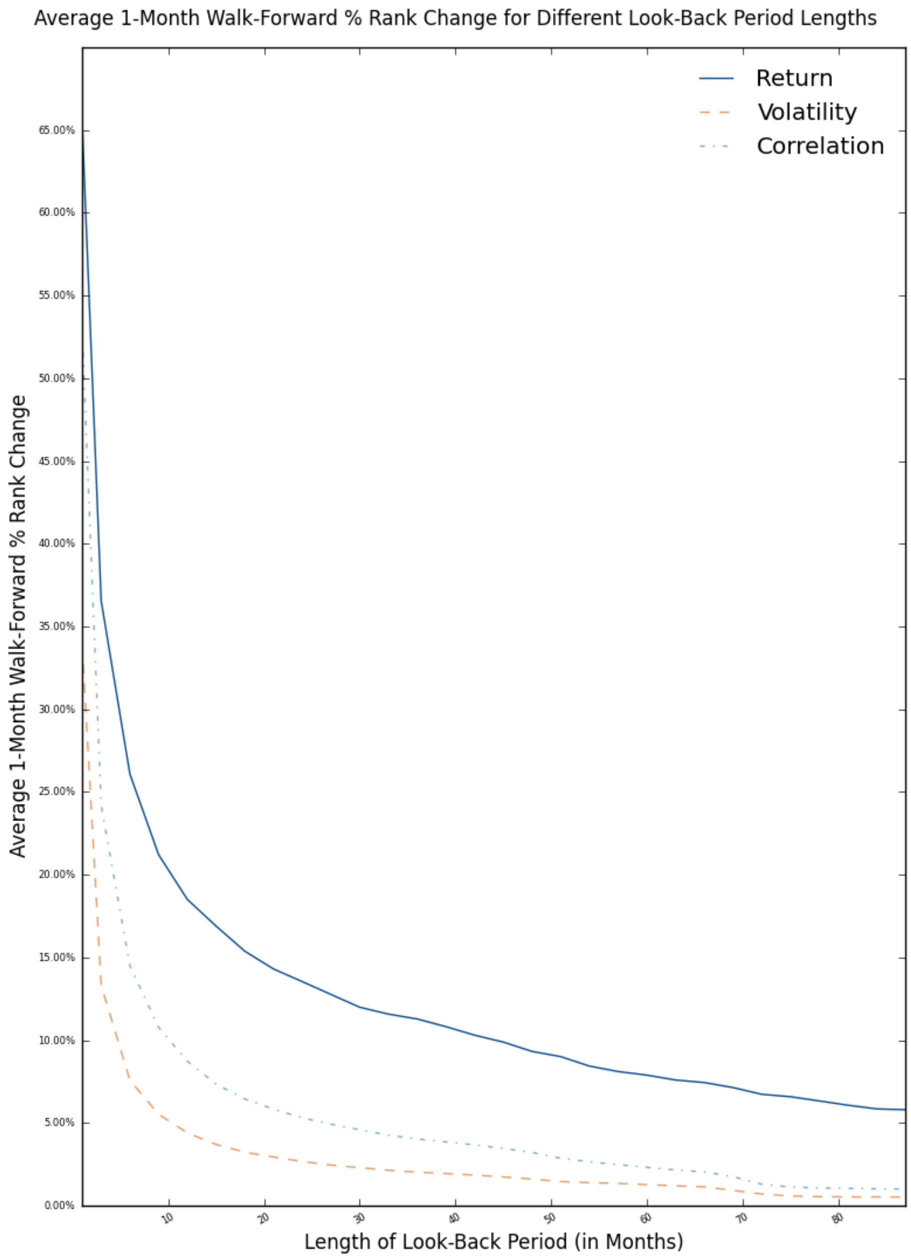

Volatilities exhibit greater stability than correlations, which exhibit greater stability than returns – which may be exactly why relying on a ranking method worked for the minimum variance test but failed in the maximum risk test. Plotted below are the mean changes with 95% confidence intervals.

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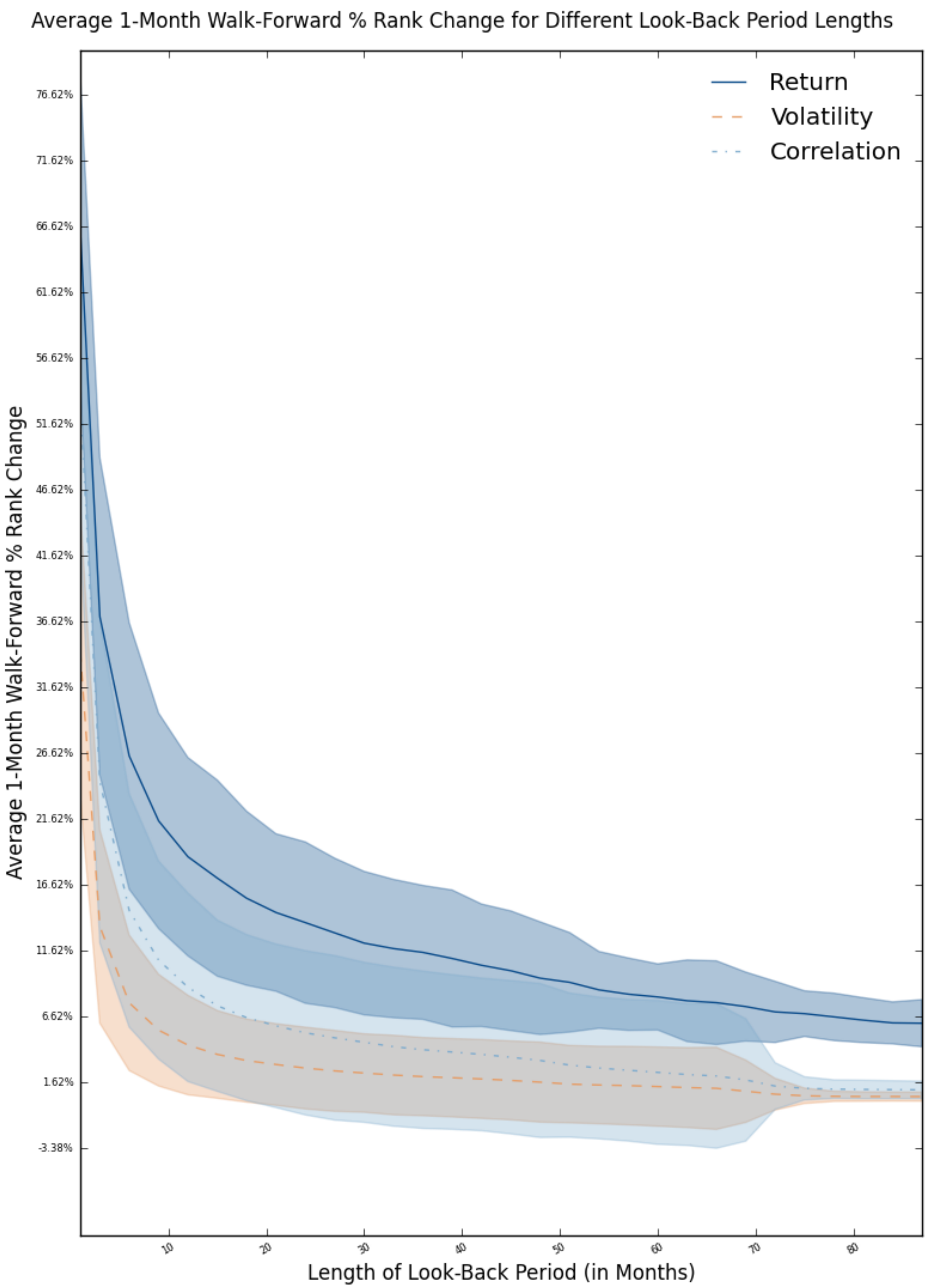

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Test Limitations

This test shares the same limitations as outlined in the Minimum Variance portfolio test case, namely a limited market environment and a limited number of walk-forward periods for long sample-length tests.

The Most Diversified Portfolio

Choueifaty (2006) proposed a measured of portfolio diversification called the Diversification Ratio, defined as the ratio of the weighted average of asset volatilities to portfolio volatility:

$$DR(w) = \frac{\Sigma_{i=1}^{N} w_i \sigma_i}{w^T \Sigma w}$$

The measure captures the impact of covariance on the reduction of volatilities in a long-only portfolio; an equal-weighted portfolio of N independent assets will have a DR of $\sqrt{N}$. Choueifaty, Froidure, and Reynier (2011) later show that DR increases in the cases of either an increase in average pairwise correlation or an increase in weight concentration.

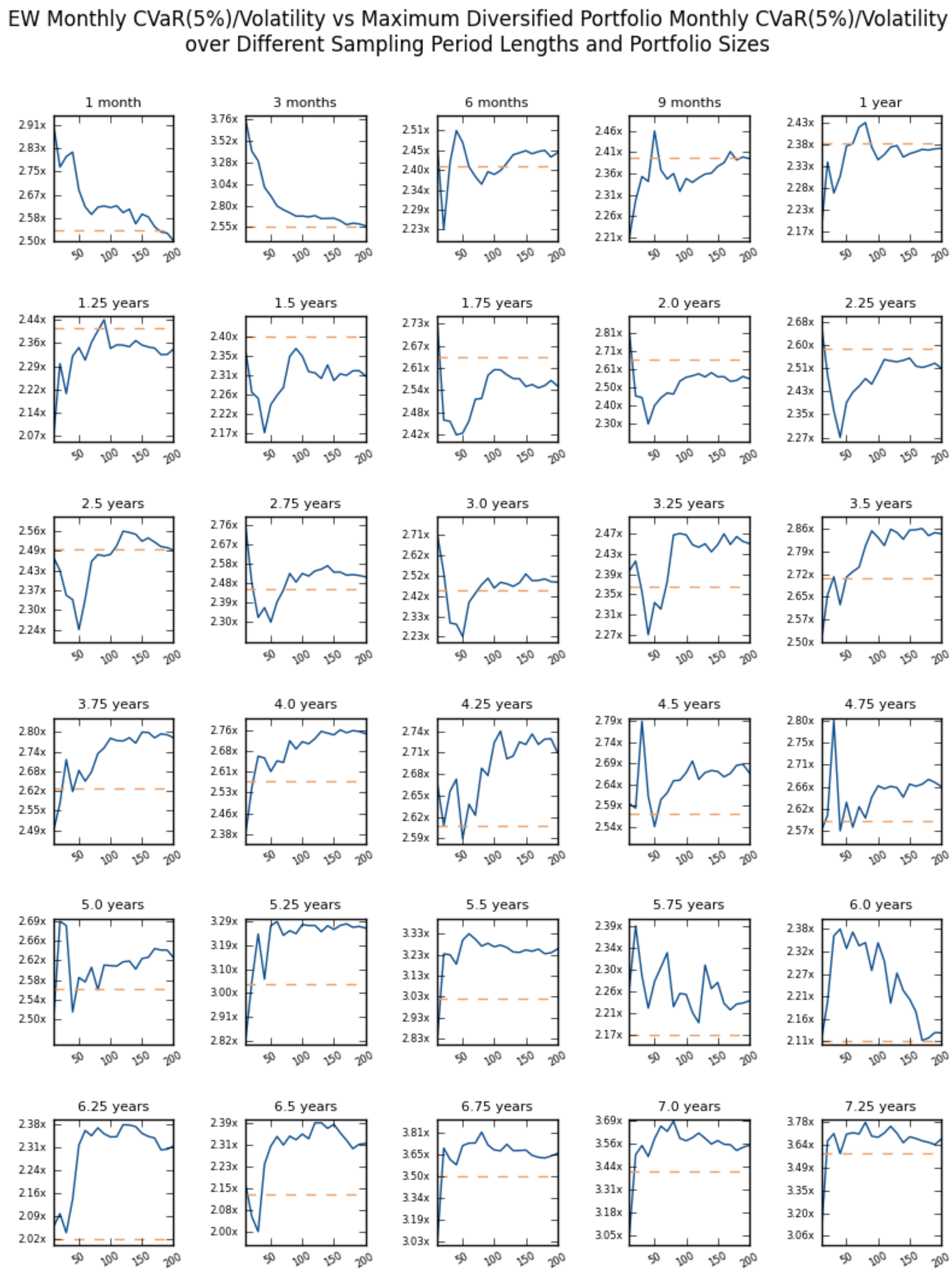
The Test

As a naïve comparison, an equal weight methodology is selected that equal-weights the N securities with the lowest average absolute pair-wise correlation.

Both portfolios rebalance monthly, using the previous M days' data to re-calibrate. The formulation of the test allows the stability of correlation ranks to be analyzed in the context of complex interaction.

1. Is there a portfolio size threshold whereby EW or MVO methods supersede each other? How does this portfolio size compare to traditional portfolio sizes held by individual investors?
2. Is there a look-back period size required for variance-covariance matrix estimate stability that provides MVO with an edge?

The comparison metric selected is 5% CVaR divided by annualized volatility (called the "convexity coefficient"); a higher value should be a proxy indicative of higher kurtosis, and therefore a breakdown in diversification. The results are plotted below. For each plot, the orange, dashed line represents the MVO convexity coefficient return using an M-month look-back period. The blue, solid line in each plot is the realized convexity coefficient for an EW portfolio made up of N assets, where N is plotted on the x-axis.



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In line with the stability plots, the naïve method only begins to provide benefit in the 1.5 – 3.25 year range for portfolios made up of approximately 50 securities. This likely represents a balance of short-term stability in correlation regimes as well as enough securities to minimize the impact of individual security risk. The other possibility is that 1.5 – 3.25 year time-periods are short enough that the more complex methodology *over-fit* historical data, making it non-robust to future changes in the market environment. For shorter periods, the simple method rapidly approaches the complex method with an increase in number of securities; it is only in longer horizon sampling periods that the more complex method beats out the simpler heuristic, likely due to the greater walk-forward rank stability of both volatility and correlation at these horizon lengths.

It would not be unfair, however, to say that this test is conclusively in favor of either strategy; for a realistic portfolio size and a couple years of data, neither methodology seems to truly trump. In which case, why go through the hassle of a more complicated method that has the risk of introducing greater error due to parameter uncertainty?

Summary

In this paper we have explored four different portfolios, all achieved through optimization techniques: the maximum Sharpe ratio portfolio, the minimum risk portfolio, the maximum expected excess return portfolio, and the maximally diversified portfolio. For each portfolio, we constructed a naïve proxy implementation and constructed a metric to compare the results of both allocation methodologies with respect to their desired goals.

For the maximum Sharpe ratio, a variety of test cases are considered. In those cases that exhibited parameter stability, mean-variance optimization did fulfill its guarantee of the most optimal Sharpe ratio portfolio in walk-forward tests; in tests where parameter stability was more uncertain, a naïve equal-weight methodology provided a better walk-forward Sharpe ratio.

In the case of the minimum variance portfolio, an equal-weight methodology made up of the N lowest-volatility securities beats a more complex optimization methodology for all test cases. Examining the rank-stability of returns, volatilities and correlations reveals that the optimization method's reliance on correlation estimates introduce parameter uncertainty that lead to sub-optimal walk-forward portfolios.

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For the maximum expected excess return portfolios, neither the naïve equal-weight methodology nor the more complex optimization methodology come out as a clear winner. This is likely due to the fact that both methods rely on stability of expected excess return estimates, which are the least stable of the parameters estimated.

Finally, the maximally diversified portfolio is considered. Again, neither methodology emerges as the clear victor, though the naïve equal-weight method seems to out-perform in portfolios of about 40-60 securities for 1.5 - 3.25 year look-back periods. This result may be more indicative of the stability of the correlation regimes of the last decade than the actual benefit of the methodology, however.

In conclusion, we find that only in cases of extreme parameter stability do more complicated allocation techniques out-perform their naïve proxies; in fact, we find that the technique that relies on the parameters with the least rank-stability underperform. When both methodologies rely on parameters with the same degree of rank-stability, the results are non-conclusive.

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